On the Brieskorn (a,b)-module of an isolated hypersurface singularity.

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Abstract

We show in this note that for a germ g of holomorphic function with an isolated singularity at the origin of $\mathbb{C}^n$ there is a pole for the meromorphic extension of the distribution

$$\frac{1}{\Gamma(\lambda)}\int_X |g|^{2\lambda}\bar{g}^{-n}\square \tag{*}$$

at $-n-\alpha$ when α is the smallest root in its class modulo $\mathbb{Z}$ of the reduce Bernstein-Sato polynomial of g. This is rather unexpected result comes from the fact that the self-duality of the Brieskorn (a,b)-module E_g associated to g exchanges the biggest simple pole sub-(a,b)-module of E_g with the saturation of E_g by $b^{-1}a$.
In the first part of this note, we prove that the biggest simple pole sub-(a,b)-module of the Briekorn (a,b)-module E of g is "geometric" in the sense that it depends only on the hypersurface germ $\{g=0\}$ at the origin in $\mathbb{C}^n$ and not on the precise choice of the reduced equation g, as the poles of (*).
By duality, we deduce the same property for the saturation $\tilde{E}$ of E. This duality gives also the relation between the "dual" Bernstein-Sato polynomial and the usual one, which is the key of the proof of the theorem.

Key words Isolated hypersurface singularity, Brieskorn (a,b)-module, Bernstein-Sato polynomial, dual Bernstein-Sato polynomial.

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1 Introduction.

Let $\tilde{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a germ of holomorphic function with an isolated singularity. Denote by $g : X \to D$ a Milnor representative of $\tilde{g}$.
Let b_g be the reduced Bernstein-Sato polynomial of g. Let α be the biggest root of b_g in its class modulo $\mathbb{Z}$. A classical question is whether for $j \in \mathbb{N}$ big enough the meromorphic extension of the distribution

$$\frac{1}{\Gamma(\lambda)} \int_X |g|^{2\lambda} \bar{g}^{-j} \square$$

has a pole at $\lambda = \alpha$.
The present note gives a result which, in a sense, suggests that, may be, this question is not the good one.
Let me introduce the dual Bernstein-Sato polynomial of g by the formula

$$b_g^*(z) = (-1)^q . b_g(-n - z)$$

where $q := deg(b_g)$. Recall that all roots of b_g (and b_g^*) are contained in $]-n, 0[$, see [K.76] for the inequality < 0, and the section 3 for the inequality $> -n$.

We shall prove the following result.

Théorème 1.0.1 *Let α be the <u>smallest</u>[1] root of b_g in its class modulo $\mathbb{Z}$, and let d be its multiplicity (as a root of b_g). Then the meromorphic extension of the of the distribution*

$$\frac{1}{\Gamma(\lambda)} \int_X |g|^{2\lambda} \bar{g}^{-n} \square$$

has a pôle of order $\geq d$ at $-n - \alpha$.

Remarks.

1. In general $b_g^* \neq b_g$ so it is not clear that $-n - \alpha$ is a root of b_g. But, of course, the previous theorem implies that there exists at least d roots of b_g (counting multiplicities) which are bigger than $-n - \alpha$. If $-n - \alpha \in [-1, 0[$ then there is no choice : $-n - \alpha$ is a root of multiplicity $\geq d$ of b_g.

2. This result gives, in term of the Bernstein-Sato polynomial b_g, a precise value where we know that a pole appears in the class $[\beta]$ modulo $\mathbb{Z}$ of a root β of b_g. But the pole which is given is not at the biggest root of b_g in this class but at the biggest root of b_g^* in this class !
   A clear reason for that is given in the proof: the dual Bernstein-Sato polynomial is the minimal polynomial of $-b^{-1}a$ acting on $F/b.F$ where F is the biggest simple pole sub-(a,b)-module of the Brieskorn (a,b)-module E associated to g. So it lies in the lattice given by <u>holomorphic</u> forms.

[1] recall that we are dealing with negative numbers.

On the contrary, b_g is the the minimal polynomial of $-b^{-1}a$ acting on $\tilde{E}/b\tilde{E}$ where $\tilde{E}$ is the saturation of E by $b^{-1}a$, or, in other words, the minimal simple pole (a,b)-module containing E. So, if E is not a simple pole (a,b)-module, elements in $\tilde{E}$ are not always representable in the holomorphic lattice, and so we may need some power of g as denominators. And this may introduce integral shifts for the poles.

3. The case where E is a simple pole (a,b)-module (that is to say when we have $F = E = \tilde{E}$) corresponds to a quasi-homogeneous g, with a suitable choice of coordinates. In this case we have $b_g^* = b_g$, so $-n - \alpha$ is the smallest root of b_g in its class modulo $\mathbb{Z}$.

In the first part of this note, we prove that the biggest simple pole sub-(a,b)-module of the Briekorn (a,b)-module E of g is "geometric" in the sense that it depends only on the hypersurface germ $\{g = 0\}$ at the origin in $\mathbb{C}^n$ and not on the precise choice of the reduced equation g.
Remark that the poles of the meromorphic distributions $\frac{1}{\Gamma(\lambda)}\int_X |g|^{2\lambda}\bar{g}^{-j}\square$ are also "geometric" in the sense above.
By duality, we deduce the same property for the saturation $\tilde{E}$ of E. This duality gives also the relation between the dual Bernstein-Sato polynomial and the usual one, which is the key of the proof of the theorem.

2 Changing the reduced equation.

Let $g : X \to D$ be a Milnor representative of a germ of an holomorphic function with an isolated singularity at the origin of $\mathbb{C}^n, n \geq 2$. We define the function

$$f(t,x) := e^t.g(x) \quad where \quad f : \mathbb{C} \times X \to \mathbb{C}$$

and we denote by $\pi : \mathbb{C} \times \mathbb{C} \times X$ the projection defined by $\pi(\lambda, t, x) = (t, x)$. We shall denote by F the function $\pi^*(f)$. Its critical locus is $S := \mathbb{C} \times \mathbb{C} \times \{0\}$.
We consider on $Y = \{F = 0\}$, as in [B.05], the complex of sheaves $\big((\hat{\mathcal{K}}er\, dF)^\bullet, d^\bullet\big)$.
The following theorem is an easy generalization of [B.05] th.2.2 (case LII) .

Théorème 2.0.2 *In the situation describe above, the $n-th$ cohomology sheaf of the complex $\big((\hat{\mathcal{K}}er\, dF)^\bullet, d^\bullet\big)$ is a constant sheaf whose fiber is F_g the biggest simple pole sub-(a,b)-module of the Brieskorn (a,b)-module E_g associated to the function g.*

It is easy to deduce from the previous theorem the following corollary.

Corollaire 2.0.3 *Let g be a germ of an holomorphic function with an isolated singularity at the origin of $\mathbb{C}^n$. Let h be any invertible holomorphic germ at the origin. Then the biggest simple pole sub-(a,b)-module of the Brieskorn (a,b)-module associated to the function $h.g$ does not depend on the choice of h up to isomorphism.*

More precisely, if the holomorphic invertible function depends holomorphically on some parameter λ in a complex manifold Λ, the subsheaf of the sheaf on Λ defined by the Brieskorn (a,b)-modules of the fibers[2]*, which is given in each fiber by the biggest simple pole sub-(a,b)-module of the Brieskorn (a,b)-module, is a locally constant sheaf on Λ.*

Proof of the theorem. Let us first consider the case of an holomorphic function f on a complex manifold Z and let the holomorphic function F be $F := \pi^*(f)$ on $\mathbb{C} \times Z$ where $\pi : \mathbb{C} \times Z \to Z$ is the projection.
In this situation we have the following description of $(\hat{\mathcal{K}}er\, dF)^p$:

$$(\hat{\mathcal{K}}er\, dF)^p = \pi^*((\hat{\mathcal{K}}er\, df)^p) \oplus d\lambda \wedge \pi^*((\hat{\mathcal{K}}er\, df)^{p-1}).$$

Then $\alpha \oplus d\lambda \wedge \beta \in (\hat{\mathcal{K}}er\, dF)^p$ is $d-$closed iff it satisfies :

$$d_{/}\alpha = 0 \quad \text{and} \quad \frac{\partial\alpha}{\partial\lambda} = d_{/}\beta$$

where $\frac{\partial\alpha}{\partial\lambda}$ is defined by the equation $d\alpha = d_{/}\alpha + d\lambda \wedge \frac{\partial\alpha}{\partial\lambda}$.

Lemme 2.0.4 *In the situation above set $Y = \{f = 0\}$; we have the short exact sequence of complex of sheaves on $\mathbb{C} \times Y$:*

$$0 \to (\hat{\mathcal{K}}er\, dF^\bullet, d^\bullet) \to \big(\pi^*(\hat{\mathcal{K}}er\, df^\bullet), d_{/}^\bullet\big) \xrightarrow{\frac{\partial}{\partial\lambda}} \big(\pi^*(\hat{\mathcal{K}}er\, df^\bullet), d_{/}^\bullet\big) \to 0.$$

So if the sheaf $\hat{\mathcal{H}}_f^{p-1}$ is 0 on Z for $p \geq 3$ or is isomorphic to[3] *$E_1 \otimes \mathbb{C}_Y$ for $p = 2$, then we have for $p \geq 2$ the exact sequence of sheaves on $\mathbb{C} \times Y$:*

$$0 \to \hat{\mathcal{H}}_F^p \xrightarrow{i} \pi^*(\hat{\mathcal{H}}_f^p) \xrightarrow{\partial/\partial\lambda} \pi^*(\hat{\mathcal{H}}_f^p).$$

Proof. Here the sheaf $\pi^*(\hat{\mathcal{H}}_f^p)$ is defined via $\lambda-$relative holomorphic forms. On this complex we have a derivation $\partial/\partial\lambda$ commuting with the product by the function F, the wedge product with dF and the $\lambda-$relative de Rham differential denoted by $d_{/}$. Remark also that we have $d_{/}F = dF$.
The exactness of the short exact sequence of complexes is obvious and the associated long exact cohomology sequence is enough to conclude for $p \geq 3$. For the $p = 2$ case, we have only to check the injectivity of the map i.
Let $\alpha \oplus d\lambda \wedge \beta \in (\hat{\mathcal{K}}er\, dF)^p \cap Ker\, d$; its image by i is the class $[\alpha]$. If it vanishes

[2] we defini this sheaf via the cohomology of the formal completion of the de Rham complex of $\Lambda-$relative holomorphic forms annihilated by $\wedge dF$.

[3] recall that $E_1 := \mathbb{C}[[b]].e_1$ where $a.e_1 = b.e_1$.

in $\pi^*(\hat{\mathcal{H}}^p_f)$ we can find $\gamma \in \pi^*((\hat{\mathcal{K}}er\, df)^{p-1})$ such that $d_/\gamma = \alpha$. Differentiating with respect to λ gives, using the relation $\frac{\partial\alpha}{\partial\lambda} = d_/\beta$,

$$d_/(\beta - \frac{\partial\gamma}{\partial\lambda}) = 0.$$

But as $\beta - \frac{\partial\gamma}{\partial\lambda} \in \pi^*((Ker\, df)^{p-1})$ this form induces a class in $\pi^*(\hat{\mathcal{H}}^{p-1}_f)$. So we can write

$$\beta = \frac{\partial\gamma}{\partial\lambda} + \varphi(\lambda, f).df$$

where $\varphi \in \pi^*(\mathbb{C}[[z]])$. We obtain, if $\frac{\partial\psi}{\partial\lambda}(\lambda, f) = \varphi(\lambda, f)$:

$$\alpha + d\lambda \wedge \beta = d(\gamma + \psi(\lambda, f).df)$$

which allows to conclude, as $\gamma + \psi(\lambda, f).df$ is in $\pi^*((\hat{\mathcal{K}}er\, df)^1)$. ■

End of the proof of the theorem. We proved in [B-05] theorem 2.2 that the sheaf $\hat{\mathcal{H}}^n_f$ is a constant sheaf on $\mathbb{C} \times \{0\} \subset \mathbb{C} \times X = Z$ with fiber the biggest simple pole sub-(a,b)-module in E_g. So the sams is true for the sheaf $\hat{\mathcal{H}}^n_F$ on $\mathbb{C} \times \mathbb{C} \times \{0\}$.■

Proof of the corollary. Let $c : \mathbb{C} \times X \to \mathbb{C}$ be an holomorphic function and set $h_\lambda(x) := exp(c(\lambda, x))$ for $\lambda \in \mathbb{C}$ and $x \in X$. Choose the following coordinate system on $\mathbb{C} \times \mathbb{C} \times X$ near the point $(\lambda_0, t_0, 0)$:

$$\lambda' = \lambda, \quad t' = t - c(\lambda, x), \quad x' = x.$$

Then the function F is transformed in $\tilde{F}(\lambda', t', x') = e^{t'}.(e^{c(\lambda', x')}.g(x')) = F(\lambda, t, x)$. The corollary follows, because we can always join two invertible functions inside an holomorphic family of invertible functions (and the restriction of a constant sheaf is a constant sheaf). ■

3 The dual Bernstein-Sato polynomial.

We shall now consider an (a,b)-module E such that

i) The (a,b)-module E is regular (see [B.93]).

ii) There exists a complex number δ and an isomorphism of (a,b)-modules $\kappa : \check{E} \to Hom_{a,b}(E, E_\delta)$, where $\check{E}$ is the (a,b)-module E in which "a" and "b" acts as $-a$ and $-b$.

Recall, for the convenience of the reader, that E_δ is the rank 1 (a,b)-module defined by $E_\delta := \mathbb{C}[[b]].e_\delta$ where a acts by $a.e_\delta = \delta.b.e_\delta$.
If E and F are (a,b)-modules, the (a,b)-module $Hom_{a,b}(E, F)$ is defined as

follows : we define on the $\mathbb{C}[[b]]-$module $Hom_{\mathbb{C}[[b]]}(E,F)$, which is free and of finite rank, an action of a by the formula :

$$(a.\varphi)(x) = a_F.\varphi(x) - \varphi(a_E.x), \quad \forall x \in E.$$

Of course, we have to check that $a.\varphi$, defined in this way, is $\mathbb{C}[[b]]-$linear and that we have $a.b.\varphi - b.a.\varphi = b^2.\varphi$. It is not difficult to check also that $Hom_{a,b}(E,F)$ is regular when E and F are regular (see [B.95]).
Recall also that the Brieskorn (a,b)-module of a germ of holomorphic function with an isolated singularity in $\mathbb{C}^n$ satisfies properties i) and ii) above with $\delta = n$, see [Be.01].

Proposition 3.0.5 *Under hypotheses i) and ii) above, let F be the biggest simple pole sub-(a,b)-module in E, and let $\tilde{E}$ the saturation of E for $b^{-1}a$.*
Then we have natural isomorphisms of (a,b)-modules deduced from κ :

$$\kappa' : \check{\tilde{E}} \to Hom_{a,b}(F, E_\delta) \quad \text{and} \quad \kappa'' : \check{F} \to Hom_{a,b}(\tilde{E}, E_\delta).$$

In the proof of this proposition we shall use the following lemmas.

Lemme 3.0.6 *Let E and F be simple pole (a,b)-modules. Then $Hom_{a,b}(E,F)$ is also a simple pole (a,b)-module.*

Proof. Fix an element $\varphi \in Hom_{a,b}(E,F)$. Then define $\theta : E \to F$ by the formula $\theta(x) := b^{-1}.a.\varphi(x) - b^{-1}.\varphi(a.x)$ for all $x \in E$. As E has a simple pole, we have $a.x \in b.E$ and so $\varphi(a.x) \in b.F$ from $b-$linearity of φ. But F has also a simple pole, so $b^{-1}.a : F \to F$ is well defined.
Now θ is $b-$linear :

$$\begin{aligned}\theta(b.y) &= b^{-1}.a.\varphi(b.y) - b^{-1}.\varphi(a.b.y) = (a+b).\varphi(y) - \varphi((a+b).y) \\ &= a.\varphi(y) - \varphi(a.y) = b.\theta(y).\end{aligned}$$

But we have $a.\varphi = b.\theta$ in $Hom_{a,b}(E,F)$. Therefore $Hom_{a,b}(E,F)$ is a simple pole (a,b)-module. ■

Lemme 3.0.7 *Let E be a regular (a,b)-module and let δ be any complex number. Then we have a canonical (a,b)-module isomorphism*

$$\tau : E \to Hom_{a,b}(Hom_{a,b}(E, E_\delta), E_\delta).$$

Proof. The map τ is defined by $x \to \tau(x)[\varphi] = \varphi(x)$. It is obviously a $b-$linear isomorphism. So we have only to check the $a-$linearity. But, with the notation $\theta = \tau(x)$, we have :

$$(a.\theta)[\varphi] = a.\big(\theta[\varphi]\big) - \theta[a.\varphi] = a.\varphi(x) - \big(a.\varphi(x) - \varphi(a.x)\big) = \tau(a.x)[\varphi].$$

And so $a.\tau(x) = \tau(a.x)$. ■

Lemme 3.0.8 *Let E and F be two (a,b)-modules. Then we have a canonical isomorphism*

$$Hom_{a,b}(E,F)\check{} \to Hom_{a,b}(\check{E},\check{F}).$$

Proof. It is clear that $Hom_{a,b}(\check{E},\check{F})$ is the same complexe vector space than $Hom_{a,b}(E,F)$ and that the action of b on it is given by $-b$. The fact that the action of a is the opposite of the action of a on $Hom_{a,b}(E,F)$ follows also directly from the definition of $Hom_{a,b}$. ■

Proof of proposition 3.0.5. The functor $Hom_{a,b}(-,E_\delta)$ applied to the inclusion of E in $\tilde{E}$ gives an (a,b)-linear injection

$$Hom_{a,b}(\tilde{E},E_\delta) \hookrightarrow Hom_{a,b}(E,E_\delta) \simeq \check{E}.$$

As $Hom_{a,b}(\tilde{E},E_\delta)$ has a simple pole by lemma 3.0.6 it is contained in $\check{F}$, by definition of F. Apply now the functor $Hom_{a,b}(-,E_\delta)$ to the inclusions

$$Hom_{a,b}(\tilde{E},E_\delta) \hookrightarrow \check{F} \hookrightarrow \check{E}$$

This gives (a,b)-linear injections

$$Hom_{a,b}(\check{E},E_\delta) \hookrightarrow Hom_{a,b}(\check{F},E_\delta) \hookrightarrow \tilde{E}$$

using lemma 3.0.7. But, as $\check{E}_\delta$ is canonically isomorphic to E_δ, so we have isomorphims

$$Hom_{a,b}(\check{E},E_\delta) \simeq Hom_{a,b}(\check{E},\check{E}_\delta) \simeq Hom_{a,b}(E,E_\delta)\check{} \simeq \check{\check{E}} \simeq E$$

using lemma 3.0.8 and our hypothesis on E. So the simple pole (a,b)-module $Hom_{a,b}(\check{F},E_\delta)$ which lies between E and $\tilde{E}$ is equal to $\tilde{E}$. We conclude using again the canonical isomorphism between E_δ and $\check{E}_\delta$ and the lemma 3.0.7. ■

Remark.

In the situation of the proposition 3.0.5 the non-degenerate (a,b)-bilinear pairing

$$h : \check{E} \times E \to E_\delta$$

deduced from κ via the formula $h(x,y) := \kappa(x)[y]$, gives also non-degenerate (a,b)-bilinear pairings

$$h' : \check{\check{E}} \times F \to E_\delta \quad \text{and} \quad h'' : \check{F} \times \tilde{E} \to E_\delta$$

deduced from κ' and κ'' via the formulas $h'(x,y) := \kappa'(x)[y]$ and $h''(u,v) = \kappa''(u)[v]$.

An obvious consequence of proposition 3.0.5 is the following corollary of the theorem 2.0.2.

Corollaire 3.0.9 *Let g be a germ of an holomorphic function having an isolated singularity at the origin in $\mathbb{C}^n$ where $n \geq 2$. For any holomorphic invertible germ h at the origin, the saturation by $b^{-1}a$ of the Brieskorn (a,b)-module of the germ $h.g$ is independant, up to an isomorphism of (a,b)-module, of the choice of h.*
If the invertible h depends holomorphically of a parameter λ in a complex manifold Λ, the sheaf on Λ defined by the saturations of the Brieskorn (a,b)-modules of the germs $h_\lambda.g$ is a locally constant sheaf on Λ.

4 Poles of $\int_X |g|^{2.\lambda}\square$.

We shall begin by a simple definition.

Définition 4.0.10 *Let E be a regular (a,b)-module. We shall call* **dual Bernstein polynomial** *of E, denoted by b^*_E, the minimal polynomial of the linear endomorphism $-b^{-1}.a$ acting on the (finite dimensional) vector space $F/b.F$ where F is the biggest simple pole sub-(a,b)-module of E.*

Recall that the Bernstein-Sato polynimial of E is the minimal polynomial of the action of $-b^{-1}.a$ on the (finite dimensional) vector space $\tilde{E}/b.\tilde{E}$, where $\tilde{E}$, as before, is the saturation of E by $b^{-1}.a$. In other words, $\tilde{E}$ is the smallest simple pole (a,b)-module which contains E. This can be understood in two ways. Either you look in $E[b^{-1}]$ for the smallest simple pole (a,b)-module containing E. The other way is to consider the inclusion $E \to \tilde{E}$ as the initial element for inclusions of E in simple poles (a,b)-modules.

Remark.

Let δ a given complex number, and assume that the (a,b)-module E is equipped with an (a,b)-linear isomorphism

$$\kappa : \check{E} \to Hom_{a,b}(E, E_\delta).$$

Then we have $b^*_E(z) = (-1)^r.b_E(-\delta - z)$ where $r := deg(b_E)$, since $b^{-1}a$ acts on the same way on E and $\check{E}$.
So, for the Brieskorn (a,b)-module of a germ of an holomorphic function g with an isolated singularity at the origin of $\mathbb{C}^n$ the dual Bernstein polynomial is given by

$$b^*_g(z) = (-1)^r b_g(-n - z).$$

Using Malgrange positivity theorem it is easy to show that the roots of b^*_g are strictly negative. This gives, using [K.76], the fact that the roots of b_g are contained in $]-n, 0[$.

Proof of the theorem 1.0.1 The only new point for this proof, compared to [B.84 a] and [B.84 b], is the following :
In a simple pole (a,b)-module F, if a spectral value β of multiplicity d for the action of $b^{-1}.a$ on F/bF, is minimal in its class modulo $\mathbb{Z}$, there exists elements $e_1, \cdots, e_d$ in F, giving a Jordan block of size d for $b^{-1}a$ acting on F/bF, and such that they satisfy in F the relations

$$a.e_j = \beta.b.e_j + b.e_{j-1}, \ \forall j \in [1, d]$$

with the convention $e_0 = 0$ (see [B.93]).
This enable us, using the standard technics of [B.84 a], to build up $(n-1)-$holomorphic forms $\omega_1, \cdots, \omega_d$ in a neighbourghood of the origin in $\mathbb{C}^n$, such that

$$d\omega_j = \beta.\frac{dg}{g} \wedge \omega_j + \frac{dg}{g} \wedge \omega_{j-1}, \forall j \in [1, d]$$

with the convention $\omega_0 = 0$, which induce a Jordan block of size d in $H^{n-1}(F, \mathbb{C})$ where F is the Milnor fiber of g, for the eigenvalue $exp(-2i\pi.\beta)$ of the monodromy. So we avoid in this way the integral shifts coming from the use of a lattice which may be not contained in the one given by holomorphic forms and we can realize the pole of our statement for $\lambda = -\beta$, using the same strategy than in [B.84a] for eigenvalues $\neq 1$ and [B.84 b] for the eigenvalue 1. ■

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